

ALGEBRAIC CONSTRUCTION OF MULTI-SPECIES q -BOSON SYSTEM

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ABSTRACT. We construct a stochastic particle system which is a multi-species version of the q -Boson system due to Sasamoto and Wadati. Its transition rate matrix is obtained from a representation of a deformation of the affine Hecke algebra of type GL .

1. INTRODUCTION

In this article we construct a multi-species version of the q -Boson system due to Sasamoto and Wadati [9] by using a representation of a deformation of the affine Hecke algebra.

The q -Boson system is a stochastic particle system on the one-dimensional lattice \mathbb{Z} . The particles can occupy the same site simultaneously, and one particle may move from site i to $i - 1$ independently for each $i \in \mathbb{Z}$. The rate at which one particle moves from a cluster with n particles is given by $1 - q^n$, where q is a parameter of the model.

The multi-species version which we propose in this paper is described as follows. Fix a positive integer N . Each bosonic particle is colored with a positive integer which is less than or equal to N . One particle may move to the left in the same way as the q -Boson system, but the rate is different. Let $b \in \{1, 2, \dots, N\}$ be the color of the moving particle and m_j ($j = b, b + 1, \dots, N$) the number of particles with color j in the cluster from which the moving particle leaves. Then the rate is given by

$$\frac{1 - q^{m_b}}{1 - q} q^{\sum_{j=b+1}^N m_j}.$$

If $N = 1$, the transition rate matrix is equal to that of the q -Boson system up to constant multiplication.

In the following we describe how the multi-species model arises from representation theory. Hereafter we fix a positive integer k which signifies the number of particles.

In a previous paper [10], we introduced a deformation of the affine Hecke algebra of type GL_k with four parameters and found that an integrable stochastic particle system can be constructed from its representation as follows. The deformed algebra has a representation on the vector space $F(L)$ of \mathbb{C} -valued functions on the k -dimensional orthogonal lattice $L := \mathbb{Z}^k$. It is defined in terms of a generalization of the discrete integral-reflection operators due to van Diejen and Emsiz [4]. Using

them we define an operator $G : F(L) \rightarrow F(L)$, which is a discrete analogue of the propagation operator introduced by Gutkin [6] in order to construct eigenfunctions of the Hamiltonian with delta potential, and determine the discrete Hamiltonian H by the property $HG = G \sum_{i=1}^k t_i$, where t_i is the shift operator $(t_i f)(x_1, \dots, x_k) := f(\dots, x_i - 1, \dots)$. Then the operator H leaves the subspace $F(L)^{\mathfrak{S}_k}$ of symmetric functions invariant. Specializing the parameters and restricting H to $F(L)^{\mathfrak{S}_k}$, we obtain the transition rate matrix of a continuous-time Markov process. The resulting system is a continuous-time limit of the q -Hahn system due to Povolotsky [8, 1].

In this paper we generalize the above construction in a similar way to the generalization of the periodic delta Bose gas due to Emsiz, Opdam and Stokman [5]. In this article we only consider the case where one of the parameters of the deformed algebra is equal to zero. Then the algebra, which we denote by \mathcal{A}_k , essentially has two parameters α and q . The algebra \mathcal{A}_k contains the Hecke algebra \mathcal{H}_k of type A_{k-1} as a subalgebra. Let M be a left \mathcal{H}_k -module and denote by $F(L, M)$ the vector space of functions on L taking values in M . Then we can define an action of \mathcal{A}_k on $F(L, M)$, introduce the propagation operator G and determine the discrete Hamiltonian H acting on $F(L, M)$ from the property $HG = G \sum_{i=1}^k t_i$. Then the Hamiltonian H leaves a subspace $F_0(L, M)$ (see Definition 3.4 below) of $F(L, M)$ invariant.

Now the multi-species version of the q -Boson system is constructed as follows. Let U be an N -dimensional vector space. We regard $U^{\otimes k}$ as a left \mathcal{H}_k -module with respect to the action found by Jimbo [7]. Then the invariant subspace $F_0(L, U^{\otimes k})$ is identified with the vector space of functions $F(\mathcal{S})$ on the set of configurations of k bosonic particles of N species on the one-dimensional lattice \mathbb{Z} . Setting $\alpha = -(1-q)$ and restricting the Hamiltonian H to $F(\mathcal{S})$, we obtain the transition rate matrix of the multi-species q -Boson system up to an additive constant.

Using the propagation operator $G : F(L, M) \rightarrow F(L, M)$, we can construct eigenfunctions of the discrete Hamiltonian H by means of the Bethe ansatz method, which we call the Bethe wave functions. We should construct Plancherel theory for them to analyze the multi-species q -Boson system in a similar manner to the work of Borodin, Corwin, Petrov and Sasamoto [2, 3]. We leave it as a future problem.

The paper is organized as follows. In Section 2 we introduce the deformation of the affine Hecke algebra and its representation defined by the discrete integral-reflection operators. In Section 3 we define the propagation operator and the discrete Hamiltonian, and construct the Bethe wave functions. In Section 4 we derive the transition rate matrix of the multi-species q -Boson system from the discrete Hamiltonian.

2. A DEFORMATION OF THE AFFINE HECKE ALGEBRA AND ITS REPRESENTATION

2.1. Preliminaries. Throughout this paper we fix an integer $k \geq 2$. Let $V := \bigoplus_{i=1}^k \mathbb{R}v_i$ be the k -dimensional Euclidean space with the orthonormal basis $\{v_i\}_{i=1}^k$.

We denote by V^* the linear dual of V and by $\{\epsilon_i\}_{i=1}^k$ the dual basis of V^* corresponding to $\{v_i\}_{i=1}^k$. For $i, j = 1, \dots, k$ we set $\alpha_{ij} := \epsilon_i - \epsilon_j$. Then the set $R := \{\alpha_{ij} \mid i \neq j\}$ forms the root system of type A_{k-1} with the simple roots $a_i := \alpha_{i,i+1}$ ($1 \leq i < k$). We denote the set of the associated positive and negative roots by R^+ and R^- , respectively.

Let $s_i : V \rightarrow V$ ($1 \leq i < k$) be the orthogonal reflection

$$s_i(v) := v - a_i(v)a_i^\vee,$$

where $a_i^\vee := v_i - v_{i+1}$ is the simple coroot. The Weyl group W of type A_{k-1} is generated by the simple reflections $\{s_i\}_{i=1}^{k-1}$. We denote the length of $w \in W$ by $\ell(w)$. The dual space V^* is a W -module by $(w\lambda)(v) := \lambda(w^{-1}v)$ ($w \in W, \lambda \in V^*, v \in V$).

Let $v \in V$. The orbit Wv intersects the closure of the fundamental chamber

$$\overline{C_+} := \{v \in V \mid a_i(v) \geq 0 \text{ } (i = 1, \dots, k-1)\}$$

at one point. Take the shortest element $w \in W$ such that $wv \in \overline{C_+}$. We denote it by w_v . Set

$$I(v) := \{a \in R^+ \mid a(v) < 0\}.$$

If $w_v = s_{i_1} \cdots s_{i_\ell}$ is a reduced expression, then $I(v) = \{s_{i_\ell} \cdots s_{i_{p+1}}(a_{i_p})\}_{p=1}^\ell$. Therefore $\ell(w_v) = \#I(v)$ and $I(v) = R^+ \cap w_v^{-1}R^-$. From the above facts, we obtain the following lemma.

Lemma 2.1. *Suppose that $I(v_1) \subset I(v_2)$. Then $w_{v_2} = w_{w_{v_1}v_2}w_{v_1}$ and $\ell(w_{v_2}) = \ell(w_{w_{v_1}v_2}) + \ell(w_{v_1})$.*

We denote by L the k -dimensional orthogonal lattice in V defined by

$$L = \bigoplus_{i=1}^k \mathbb{Z}v_i,$$

and set

$$L_+ := L \cap \overline{C_+}.$$

Hereafter we set $\epsilon_{k+1}(x) = -\infty$ for $x \in L_+$.

For $x \in L$ and $1 \leq i \leq k$, we set

$$\begin{aligned} d_i^+(x) &:= \#\{p \mid i < p \leq k, \epsilon_i(x) = \epsilon_p(x)\}, \\ d_i^-(x) &:= \#\{p \mid 1 \leq i < p, \epsilon_i(x) = \epsilon_p(x)\}. \end{aligned}$$

We denote by $\sigma_x \in \mathfrak{S}_k$ the permutation determined by $w_x v_i = v_{\sigma_x(i)}$ ($1 \leq i \leq k$). Then it holds that

$$(2.1) \quad d_i^\pm(x) = d_{\sigma_x(i)}^\pm(w_x x), \quad \epsilon_i(x) = \epsilon_{\sigma_x(i)}(w_x x)$$

for $x \in L$ and $1 \leq i \leq k$.

Proposition 2.2. *Suppose that $1 \leq i < k$.*

- (i) If $v \in V$ satisfies $a_i(v) > 0$, then $w_{s_i v} = w_v s_i$ and $\ell(w_{s_i v}) = \ell(w_v) + 1$.
- (ii) If $x \in L$ satisfies $a_i(x) = 0$, then $\sigma_x(i+1) = \sigma_x(i) + 1$, $w_x s_i = s_{\sigma_x(i)} w_x$ and $\ell(w_x s_i) = \ell(s_{\sigma_x(i)} w_x) = \ell(w_x) + 1$.

Proof. (i) From $a_i(v) > 0$ we see that $I(s_i v) = s_i I(v) \sqcup \{a_i\}$. Hence $\ell(w_{s_i v}) = \ell(w_v) + 1$. Since $w_v s_i$ moves $s_i v$ into $\overline{C_+}$, we have $w_{s_i v} = w_v s_i$.

(ii) Since $a_i(x) = 0$ and $w_x x \in L_+$, there exists an integer p such that $1 \leq p \leq k$ and $\epsilon_{\sigma_x(i)}(w_x x) = \epsilon_{\sigma_x(i+1)}(w_x x) = \epsilon_p(w_x x) > \epsilon_{p+1}(w_x x)$. Then, for $j = i$ and $i+1$, it holds that $p - \sigma_x(j) = d_{\sigma_x(j)}^+(w_x x) = d_j^+(x)$. On the other hand, $d_i^+(x) = d_{i+1}^+(x) + 1$ because $a_i(x) = 0$. Therefore

$$\sigma_x(i+1) = p - d_{i+1}^+(x) = p - d_i^+(x) + 1 = \sigma_x(i) + 1.$$

Set $z := x - v_{i+1}/2$. Since $|a(v_j/2)| \leq 1/2$ ($j = i, i+1$) for any $a \in R$, we have $I(x) \subset I(z)$ and $I(x) \subset I(s_i z)$. Moreover, $w_{s_i z} = w_z s_i$ and $\ell(w_{s_i z}) = \ell(w_z) + 1$ from (i). Therefore, using Lemma 2.1, we get

$$(2.2) \quad w_{w_x z} w_x s_i = w_{w_x s_i z} w_x, \quad \ell(w_{w_x z}) + \ell(w_x) + 1 = \ell(w_{w_x s_i z}) + \ell(w_x).$$

Note that $w_x z = w_x x - v_{\sigma_x(i+1)}/2$ and $w_x s_i z = w_x x - v_{\sigma_x(i)}/2$. Since $w_x x \in L_+$, we have

$$I(w_x z) = \{\alpha_{\sigma_x(i+1), l}\}_{l=\sigma_x(i+1)+1}^p, \quad I(w_x s_i z) = \{\alpha_{\sigma_x(i), l}\}_{l=\sigma_x(i)+1}^p.$$

Hence

$$w_{w_x z} = s_{p-1} s_{p-2} \cdots s_{\sigma_x(i+1)}, \quad w_{w_x s_i z} = s_{p-1} s_{p-2} \cdots s_{\sigma_x(i)},$$

where the right hand sides are reduced expressions. Using $\sigma_x(i+1) = \sigma_x(i) + 1$ and (2.2), we find that $w_x s_i = s_{\sigma_x(i)} w_x$ and $\ell(w_x s_i) = \ell(s_{\sigma_x(i)} w_x) = \ell(w_x) + 1$. \square

We will also use the following proposition. See Lemma 3.9 and Lemma 3.10 in [10] for the proof.

Proposition 2.3. *Suppose that $x \in L$ and $1 \leq i \leq k$. Set $y = x - v_i$. Then it holds that*

$$(2.3) \quad \begin{aligned} & s_{\sigma_y(i)-d_i^-(y)} \cdots s_{\sigma_y(i)-1} w_y = s_{\sigma_x(i)+d_i^+(x)-1} \cdots s_{\sigma_x(i)} w_x, \\ & a_j(w_x x) = 0 \quad (\sigma_x(i) \leq j < \sigma_x(i) + d_i^+(x)), \\ & a_j(w_y y) = 0 \quad (\sigma_y(i) - d_i^-(y) \leq j < \sigma_y(i)) \end{aligned}$$

and $d_i^-(y) + \ell(w_y) = d_i^+(x) + \ell(w_x)$.

2.2. A deformation of the affine Hecke algebra and its representation.

Definition 2.4. Let α and q be complex constants. We define the algebra \mathcal{A}_k to be the unital associative \mathbb{C} -algebra with the generators $X_i^{\pm 1}$ ($1 \leq i \leq k$) and

T_i ($1 \leq i < k$) satisfying the following relations:

$$\begin{aligned} (T_i - 1)(T_i + q) &= 0 \quad (1 \leq i < k), & T_i T_{i+1} T_i &= T_i T_{i+1} T_i \quad (1 \leq i \leq k-2), \\ T_i T_j &= T_j T_i \quad (|i - j| > 1), & X_i X_j &= X_j X_i \quad (i, j = 1, \dots, k), \\ X_{i+1} T_i - T_i X_i &= T_i X_{i+1} - X_i T_i = (1 - q) X_{i+1} + \alpha \quad (1 \leq i < k), \\ X_i T_j &= T_j X_i \quad (i \neq j, j+1). \end{aligned}$$

When $\alpha = 0$, the algebra \mathcal{A}_k is isomorphic to the affine Hecke algebra of type GL_k . The subalgebra generated by T_i ($1 \leq i < k$) is isomorphic to the Hecke algebra of type A_{k-1} . We denote it by \mathcal{H}_k .

For a left \mathcal{H}_k -module M , we denote by $F(L, M)$ the complex vector space of functions on L taking values in M . The Weyl group W acts on $F(L, M)$ by $(wf)(x) := f(w^{-1}x)$ ($w \in W, f \in F(L, M), x \in L$). Let \widehat{T}_i ($1 \leq i < k$) be the \mathbb{C} -linear operator on $F(L, M)$ defined by $(\widehat{T}_i f)(x) = T_i \cdot f(x)$ ($f \in F(L, M), x \in L$), where \cdot signifies the action of \mathcal{H}_k on M . Then the assignment $T_i \mapsto \widehat{T}_i$ ($1 \leq i < k$) uniquely determines an algebra homomorphism $\mathcal{H}_k \rightarrow \text{End}(F(L, M))$. It commutes with the action of W .

We identify the group algebra $\mathbb{C}[L]$ with the Laurent polynomial ring $\mathbb{C}[e^{\pm v_1}, \dots, e^{\pm v_k}]$. The Weyl group acts on $\mathbb{C}[L]$ from the right by $e^x w = e^{w^{-1}x}$ ($x \in L, w \in W$). Using this action we define the \mathbb{C} -linear map $\check{I}_j : \mathbb{C}[L] \rightarrow \mathbb{C}[L]$ ($1 \leq j < k$) by

$$\check{I}_j(P) := (P - Ps_j) \frac{\alpha e^{v_{j+1}} + 1 - q}{1 - e^{-v_j + v_{j+1}}}.$$

Consider the non-degenerate \mathbb{C} -bilinear pairing $(\cdot, \cdot) : \mathbb{C}[L] \times F(L, M) \rightarrow M$ uniquely determined by $(e^x, f) = f(x)$ ($x \in L, f \in F(L, M)$). Now we define the \mathbb{C} -linear operator $\widehat{I}_j : F(L, M) \rightarrow F(L, M)$ ($1 \leq j < k$) by the property

$$(P, \widehat{I}_j(f)) = (\check{I}_j(P), f)$$

for any $P \in \mathbb{C}[L]$. It is explicitly written as follows.

$$(2.4) \quad (\widehat{I}_j f)(x) = \begin{cases} \sum_{l=0}^{a_j(x)-1} (\alpha f(x - la_j^\vee + v_{j+1}) + (1 - q)f(x - la_j^\vee)) & (a_j(x) > 0) \\ 0 & (a_j(x) = 0) \\ - \sum_{l=1}^{-a_j(x)} (\alpha f(x + la_j^\vee + v_{j+1}) + (1 - q)f(x + la_j^\vee)) & (a_j(x) < 0) \end{cases}$$

Lemma 2.5. *The following relations hold in $\text{End}(F(L, M))$.*

- (i) $\widehat{I}_j^2 = (1 - q)\widehat{I}_j$ for $1 \leq j < k$.
- (ii) $s_i \widehat{I}_j = \widehat{I}_j s_i$ if $|i - j| \geq 2$.
- (iii) $s_j \widehat{I}_j + \widehat{I}_j s_j = (1 - q)(s_j - 1)$ for $1 \leq j < k$.

- (iv) $\widehat{I}_j s_{j+1} s_j = s_{j+1} s_j \widehat{I}_{j+1}$ and $\widehat{I}_{j+1} s_j s_{j+1} = s_j s_{j+1} \widehat{I}_j$ for $1 \leq j \leq k-2$.
(v) For $1 \leq j \leq k-2$,

$$\begin{aligned}\widehat{I}_j s_{j+1} \widehat{I}_j &= (1-q) s_{j+1} \widehat{I}_j s_{j+1} + s_{j+1} \widehat{I}_j \widehat{I}_{j+1} + \widehat{I}_{j+1} \widehat{I}_j s_{j+1}, \\ \widehat{I}_{j+1} s_j \widehat{I}_{j+1} &= (1-q) s_j \widehat{I}_{j+1} s_j + s_j \widehat{I}_{j+1} \widehat{I}_j + \widehat{I}_j \widehat{I}_{j+1} s_j.\end{aligned}$$

- (vi) $\widehat{I}_j \widehat{I}_{j+1} \widehat{I}_j + q s_j \widehat{I}_{j+1} s_j = \widehat{I}_{j+1} \widehat{I}_j \widehat{I}_{j+1} + q s_{j+1} \widehat{I}_j s_{j+1}$ for $1 \leq j < k$.

Proof. Straightforward check. \square

Proposition 2.6. *Let M be a left \mathcal{H}_k -module. For $1 \leq j \leq k$, let $t_j : F(L, M) \rightarrow F(L, M)$ be the shift operator*

$$(t_j f)(x) := f(x - v_j) \quad (f \in F(L, M), x \in L).$$

Then the assignments

$$X_j \mapsto t_j \quad (1 \leq j \leq k), \quad T_j \mapsto \widehat{T}_j s_j + \widehat{I}_j \quad (1 \leq j < k)$$

uniquely extend to a \mathbb{C} -algebra homomorphism $\rho : \mathcal{A}_k \rightarrow \text{End}(F(L, M))$.

Proof. Note that the operators \widehat{I}_j ($1 \leq j < k$) commute with \widehat{T}_j ($1 \leq j < k$), and

$$(e^{-v_j} P, f) = (P, t_j f).$$

for $P \in \mathbb{C}[L]$ and $f \in F(L, M)$. Now we can check the defining relations of \mathcal{A}_k by using Lemma 2.5 and the equality

$$\check{I}_j(e^{x-v_{j+1}}) - e^{-v_j} \check{I}_j(e^x) = ((1-q)e^{-v_{j+1}} + \alpha)e^x = \check{I}_j(e^{x-v_j}) - e^{-v_{j+1}} \check{I}_j(e^x)$$

for $1 \leq j < k$ and $x \in L$. \square

We will often use the fact below which follows from (2.4).

Proposition 2.7. *Let M be a left \mathcal{H}_k -module. Suppose that $1 \leq j < k$, $x \in L$ and $f \in F(L, M)$. If $a_j(x) = 0$, then $(\rho(T_j)f)(x) = T_j.f(x)$.*

3. DISCRETE HAMILTONIAN

3.1. Discrete Hamiltonian and Propagation operator. For $1 \leq i \leq k$ and $x \in L$, we define $T_i^{(\pm)}(x) \in \mathcal{H}_k$ by

$$\begin{aligned}T_i^{(-)}(x) &:= T_{w_x}^{-1} \left(T_{\sigma_x(i)-1}^{-1} \cdots T_{\sigma_x(i)-d_i^-(x)}^{-1} \right) \left(T_{\sigma_x(i)-d_i^-(x)}^{-1} \cdots T_{\sigma_x(i)-1}^{-1} \right) T_{w_x}, \\ T_i^{(+)}(x) &:= T_{w_x}^{-1} \left(\sum_{j=\sigma_x(i)}^{\sigma_x(i)+d_i^+(x)-1} \left(T_{\sigma_x(i)}^{-1} \cdots T_{j-1}^{-1} \right) T_j^{-1} (T_{j-1} \cdots T_{\sigma_x(i)}) \right) T_{w_x}.\end{aligned}$$

Definition 3.1. Let M be a left \mathcal{H}_k -module. We define the discrete Hamiltonian $H : F(L, M) \rightarrow F(L, M)$ by

$$(Hf)(x) := \sum_{i=1}^k q^{d_i^-(x)} T_i^{(-)}(x) \cdot \left(f(x - v_i) - \alpha T_i^{(+)}(x) \cdot f(x) \right) \quad (f \in F(L, M), x \in L),$$

where \cdot means the left action of \mathcal{H}_k on M .

Next we define the propagation operator. Let $w = s_{i_1} \cdots s_{i_l}$ be a reduced expression of $w \in W$. Then we set $T_w := T_{i_1} \cdots T_{i_l}$.

Definition 3.2. Let M be a left \mathcal{H}_k -module. We define the *propagation operator* $G : F(L, M) \rightarrow F(L, M)$ by

$$(Gf)(x) := T_{w_x}^{-1} \cdot ((\rho(T_{w_x})f)(w_x x)) \quad (f \in F(L, M), x \in L),$$

where \cdot means the action of \mathcal{H}_k on M .

Theorem 3.3. *It holds that $HG = G(\sum_{i=1}^k t_i)$. Therefore if f is an eigenfunction of $\sum_{i=1}^k t_i$, then $G(f)$ is that of the discrete Hamiltonian H with the same eigenvalue.*

Proof. Suppose that $f \in F(L, M)$ and $x \in L$. First we fix i ($1 \leq i \leq k$) and calculate $(Gf)(x - v_i)$. Set $y = x - v_i$. From Proposition 2.3, we have

$$(3.1) \quad T_{w_y}^{-1} = T_{w_x}^{-1} (T_{\sigma_x(i)}^{-1} \cdots T_{\sigma_x(i)+d_i^+(x)-1}^{-1}) (T_{\sigma_y(i)-d_i^-(y)} \cdots T_{\sigma_y(i)-1}).$$

Using Proposition 2.7, we see that

$$(Gf)(x - v_i) = T_{w_x}^{-1} (T_{\sigma_x(i)}^{-1} \cdots T_{\sigma_x(i)+d_i^+(x)-1}^{-1}) \cdot \left(\rho(T_{\sigma_x(i)+d_i^+(x)-1} \cdots T_{\sigma_x(i)}) f \right) (w_y y).$$

From (2.3) it holds that $w_y y = w_x x - v_{\sigma_x(i)+d_i^+(x)}$. Therefore

$$\begin{aligned} & (Gf)(x - v_i) \\ &= T_{w_x}^{-1} (T_{\sigma_x(i)}^{-1} \cdots T_{\sigma_x(i)+d_i^+(x)-1}^{-1}) \cdot \left(\rho(X_{\sigma_x(i)+d_i^+(x)} T_{\sigma_x(i)+d_i^+(x)-1} \cdots T_{\sigma_x(i)}) f \right) (w_x x). \end{aligned}$$

Move $X_{\sigma_x(i)+d_i^+(x)}$ to the right using the relation $X_{j+1}T_j = T_jX_j + \alpha + (1-q)X_{j+1}$, and use Proposition 2.7 to remove $\rho(T_j)$ ($\sigma_x(i) \leq j < \sigma_x(i) + d_i^+(x)$). As a result we get

$$\begin{aligned} (3.2) \quad & (Gf)(x - v_i) = T_{w_x}^{-1} \cdot (\rho(X_{\sigma_x(i)} T_{w_x}) f) (w_x x) + \alpha T_i^{(+)}(x) \cdot G(f)(x) \\ & + (1-q) \sum_{j=\sigma_x(i)}^{\sigma_x(i)+d_i^+(x)-1} T_{w_x}^{-1} (T_{\sigma_x(i)}^{-1} \cdots T_{j-1}^{-1}) T_j^{-1} (T_{j-1} \cdots T_{\sigma_x(i)}) \cdot (\rho(X_{j+1} T_{w_x}) f) (w_x x). \end{aligned}$$

Now let us calculate $(HG(f))(x)$. The set $\{1, 2, \dots, k\}$ is decomposed into a direct sum of sub-intervals J_1, \dots, J_r so that $i \in J_a$ and $j \in J_a$ if and only if $\epsilon_i(w_x x) = \epsilon_j(w_x x)$ for $a = 1, 2, \dots, r$. For an interval $J = \{l, l+1, \dots, m\}$, set

$$K_J := \sum_{i=l}^m q^{i-l} (T_{i-1}^{-1} \cdots T_l^{-1}) (T_l^{-1} \cdots T_{i-1}^{-1}) \\ \times \left\{ X_i + (1-q) \sum_{j=i}^{m-1} (T_i^{-1} \cdots T_{j-1}^{-1}) T_j^{-1} (T_{j-1} \cdots T_i) X_{j+1} \right\}$$

Change the index i in the definition of H to $\sigma_x(i)$ and rewrite $(HG(f))(x)$ using (2.1) and (3.2). Then we obtain

$$(HG(f))(x) = \sum_{a=1}^r T_{w_x}^{-1} \cdot (\rho(K_{J_a} T_{w_x}) f)(w_x x).$$

On the other hand, K_J is rewritten as

$$K_J = \sum_{i=l}^m \left\{ q^{i-l} (T_{i-1}^{-1} \cdots T_l^{-1}) (T_l^{-1} \cdots T_{i-1}^{-1}) \right. \\ \left. + (1-q) \sum_{j=l}^{i-1} q^{j-l} (T_{i-1} \cdots T_{j+1}) (T_{j-1}^{-1} \cdots T_l^{-1}) (T_l^{-1} \cdots T_{i-1}^{-1}) \right\} X_i.$$

From the above expression, we see that $K_J = \sum_{i=l}^m X_i$ using the relation $qT_j^{-1} + (1-q) = T_j$. Thus we get

$$(HG(f))(x) = T_{w_x}^{-1} \cdot \left(\rho \left(\sum_{i=1}^k X_i T_{w_x} \right) f \right) (w_x x).$$

Since $\sum_{i=1}^k X_i$ commutes with T_i ($1 \leq i < k$), the right hand side is equal to

$$T_{w_x}^{-1} \cdot \left(\rho(T_{w_x}) \rho \left(\sum_{i=1}^k X_i \right) f \right) (w_x x) = (G(\sum_{i=1}^k t_i) f)(x).$$

This completes the proof. \square

3.2. Invariant subspace.

Definition 3.4. Let M be a left \mathcal{H}_k -module. We denote by $F_0(L, M)$ the subspace of $F(L, M)$ consisting of the functions $f : L \rightarrow M$ satisfying

$$f(s_i x) = \begin{cases} T_i^{-1} \cdot f(x) & (a_i(x) \geq 0) \\ T_i \cdot f(x) & (a_i(x) < 0) \end{cases}$$

for $1 \leq i < k$ and $x \in L$.

Note that if $f \in F_0(L, M)$ then

$$(3.3) \quad f(x) = T_{w_x}^{-1} \cdot f(w_x x)$$

for $x \in L$.

In this subsection we prove the following theorem.

Theorem 3.5. *Let M be a left \mathcal{H}_k -module. Then it holds that $H(F_0(L, M)) \subset F_0(L, M)$.*

For that purpose, we rewrite the formula of Hf ($f \in F_0(L, M)$). For $x \in L_+$, we define the *cluster coordinate* (c_1, c_2, \dots, c_r) of x by the property that $\sum_{a=1}^r c_a = k$, $\epsilon_{c_1}(x) > \epsilon_{c_1+c_2}(x) > \dots > \epsilon_{c_1+\dots+c_r}(x)$ and $\epsilon_j(x) = \epsilon_{c_1+\dots+c_a}(x)$ if $c_1 + \dots + c_{a-1} < j \leq c_1 + \dots + c_a$. For example, if $k = 5$ and $x = 2v_1 + 2v_2 - v_3 - 3v_4 - 3v_5$, then the cluster coordinate of x is $(2, 1, 2)$.

Proposition 3.6. *Let M be a left \mathcal{H}_k -module. Suppose that $f \in F_0(L, M)$ and $x \in L$. Let (c_1, \dots, c_r) be the cluster coordinate of $w_x x$. Then it holds that*

$$(3.4) \quad (Hf)(x) = T_{w_x}^{-1} \sum_{a=1}^r \sum_{l=1}^{c_a} q^{c_a-l} (T_{c_1+\dots+c_{a-1}+l}^{-1} \cdots T_{c_1+\dots+c_a-1}^{-1}) \cdot f(w_x x - v_{c_1+\dots+c_a}) \\ - \frac{\alpha}{1-q} \sum_{a=1}^r (c_a - [c_a]_q) f(x),$$

where $[n]_q := (1 - q^n)/(1 - q)$ is the q -integer.

In the proof of Proposition 3.6, we use the following formula.

Lemma 3.7. *Suppose that $f \in F_0(L, M)$, $x \in L_+$ and $0 \leq p < p + c \leq k$. If $a_j(x) = 0$ for $p + 1 \leq j < p + c$, then it holds that*

$$(3.5) \quad \sum_{l=1}^c q^{l-1} (T_{p+l-1}^{-1} \cdots T_{p+2}^{-1} T_{p+1}^{-1}) \cdot f(x - v_{p+1}) \\ = \sum_{l=1}^c q^{c-l} (T_{p+l}^{-1} T_{p+l+1}^{-1} \cdots T_{p+c-1}^{-1}) \cdot f(x - v_{p+c}).$$

Proof. For $p + 1 \leq j < m \leq p + c$, it holds that $T_j \cdot f(x - v_j) = f(x - v_{j+1})$ and $T_m^{-1} \cdot f(x - v_j) = f(x - v_j)$ because $a_j(x - v_j) = -1 < 0$ and $a_m(x - v_j) = 0$. Using $qT_j^{-1} = T_j - (1 - q)$ repeatedly, we see that

$$q^{l-1} (T_{p+l-1}^{-1} \cdots T_{p+1}^{-1}) \cdot f(x - v_{p+1}) = f(x - v_{p+l}) - (1 - q) \sum_{j=1}^{l-1} q^{l-j-1} f(x - v_{p+j}).$$

Hence the left hand side of (3.5) is equal to $\sum_{l=1}^c q^{c-l} f(x - v_{p+l})$. Since $a_j(x - v_{j+1}) = 1 \geq 0$ for $p + 1 \leq j < p + c$, we have $f(x - v_{p+l}) = (T_{p+l}^{-1} \cdots T_{p+c-1}^{-1}) \cdot f(x - v_{p+c})$ for $1 \leq l \leq c$. This completes the proof. \square

Proof of Proposition 3.6. Note that $T_i \cdot f(x) = f(x)$ if $a_i(x) = 0$. Using (2.3) and (3.3), we see that

$$\sum_{i=1}^k q^{d_i^-(x)} T_i^{(-)}(x) T_i^{(+)}(x) \cdot f(x) = \sum_{i=1}^k q^{d_i^-(x)} d_i^+(x) f(x) = \frac{1}{1-q} \sum_{a=1}^r (c_a - [c_a]_q) f(x).$$

Hence it suffices to show that

$$(3.6) \quad \sum_{i=1}^k q^{d_i^-(x)} T_i^{(-)}(x) \cdot f(x - v_i) \\ = T_{w_x}^{-1} \sum_{a=1}^r \sum_{l=1}^{c_a} q^{c_a-l} (T_{c_1+\dots+c_{a-1}+l-1}^{-1} \cdots T_{c_1+\dots+c_{a-1}}^{-1}) \cdot f(w_x x - v_{c_1+\dots+c_a}).$$

Fix $1 \leq i \leq k$ and set $y = x - v_i$. From (3.1) and (3.3) we have

$$f(y) = T_{w_x}^{-1} (T_{\sigma_x(i)}^{-1} \cdots T_{\sigma_x(i)+d_i^+(x)-1}^{-1}) f(w_y y).$$

Therefore

$$T_i^{(-)}(x) \cdot f(y) = T_{w_x}^{-1} (T_{\sigma_x(i)-1}^{-1} \cdots T_{\sigma_x(i)-d_i^-(x)}^{-1}) (T_{\sigma_x(i)-d_i^-(x)}^{-1} \cdots T_{\sigma_x(i)+d_i^+(x)-1}^{-1}) \cdot f(w_y y).$$

Since $w_y y = w_x x - v_{\sigma_x(i)+d_i^+(x)}$ and $a_j(w_x x) = 0$ for $\sigma_x(i) - d_i^-(x) \leq j < \sigma_x(i) + d_i^+(x)$, the right hand side above is equal to

$$T_{w_x}^{-1} (T_{\sigma_x(i)-1}^{-1} \cdots T_{\sigma_x(i)-d_i^-(x)}^{-1}) \cdot f(w_x x - v_{\sigma_x(i)-d_i^-(x)}).$$

Note that if $c_1 + \dots + c_{a-1} < \sigma_x(i) \leq c_1 + \dots + c_a$, then $\sigma_x(i) - d_i^-(x) = c_1 + \dots + c_{a-1} + 1$, which is independent of i . Thus the left hand side of (3.6) is equal to

$$T_{w_x}^{-1} \sum_{a=1}^r \sum_{l=1}^{c_a} q^{l-1} (T_{c_1+\dots+c_{a-1}+l-1}^{-1} \cdots T_{c_1+\dots+c_{a-1}+1}^{-1}) \cdot f(w_x x - v_{c_1+\dots+c_{a-1}+1}).$$

Now the equality (3.6) is an immediate consequence of Lemma 3.7. \square

Now let us prove Theorem 3.5.

Proof of Theorem 3.5. Suppose that $f \in F_0(L, M)$, $x \in L_+$ and $1 \leq i < k$. Note that $w_{s_i x} s_i x = w_x x$ and hence the cluster coordinates of x and $s_i x$ are the same.

If $a_i(x) > 0$, then $T_{w_{s_i x}} = T_{w_x} T_i$ from Proposition 2.2. Using (3.4), we see that $(Hf)(s_i x) = T_i^{-1} \cdot (Hf)(x)$. Changing x to $s_i x$, we find that $(Hf)(s_i x) = T_i \cdot (Hf)(x)$ if $a_i(x) < 0$.

Let us consider the case where $a_i(x) = 0$. From Proposition 2.2, it holds that $T_i^{-1} T_{w_x}^{-1} = T_{w_x}^{-1} T_{\sigma_x(i)}^{-1}$. Note that $d_{\sigma_x(i)}^+(w_x x) = d_i^+(x) > 0$ because $\epsilon_{i+1}(x) = \epsilon_i(x)$. Hence there exists $1 \leq a \leq r$ such that $c_1 + \dots + c_{a-1} < \sigma_x(i) < c_1 + \dots + c_a$. Now Lemma 3.8 below implies that $T_i^{-1} \cdot (Hf)(x) = (Hf)(x)$. \square

Lemma 3.8. *Suppose that $x \in L_+$, $f \in F_0(L, M)$ and $0 \leq p < p + c \leq k$. Set*

$$J = \sum_{l=1}^c q^{c-l} (T_{p+l}^{-1} T_{p+l+1}^{-1} \cdots T_{p+c-1}^{-1}).$$

If $a_j(x) = 0$ for $p + 1 \leq j < p + c$, then

$$T_{p+i}^{-1} J \cdot f(x - v_{p+c}) = J \cdot f(x - v_{p+c})$$

for $1 \leq i \leq c - 1$.

Proof. Using the quadratic relation $q^{-1}T_j^{-2} = 1 - (1 - q)T_j^{-1}$, we have

$$\begin{aligned} T_{p+i}^{-1} J &= \sum_{l=1}^{i-1} q^{c-l} (T_{p+l}^{-1} \cdots T_{p+c-1}^{-1}) T_{p+i-1}^{-1} + \sum_{l=i, i+1} q^{c-l} (T_{p+l}^{-1} \cdots T_{p+c-1}^{-1}) \\ &\quad + \sum_{l=i+2}^c q^{c-l} (T_{p+l}^{-1} \cdots T_{p+c-1}^{-1}) T_{p+i}^{-1}. \end{aligned}$$

Note that the first and the third term in the right hand side vanish if $i = 1$ and $i = c - 1$, respectively. Since $T_j^{-1} \cdot f(x - v_{p+c}) = f(x - v_{p+c})$ for $p + 1 \leq j \leq p + c - 2$, we obtain the desired formula. \square

3.3. Bethe wave functions. Let M be a left \mathcal{H}_k -module. We construct eigenfunctions of the restriction $H|_{F_0(L, M)}$, which we call the *Bethe wave functions*.

Denote by $F(L, M)^{\mathcal{H}_k}$ the subspace of $F(L, M)$ consisting of the $\rho(\mathcal{H}_k)$ -invariant functions, that is,

$$F(L, M)^{\mathcal{H}_k} := \{f \in F(L, M) \mid \rho(T_i)f = f \text{ for } 1 \leq i < k\}.$$

Proposition 3.9. *Let M be a left \mathcal{H}_k -module. It holds that $G(F(L, M)^{\mathcal{H}_k}) \subset F_0(L, M)$. Therefore if $h \in F(L, M)^{\mathcal{H}_k}$ is an eigenfunction of $\sum_{i=1}^k t_i$, then $G(h)$ is that of the operator $H|_{F_0(L, M)}$ with the same eigenvalue.*

Proof. Let f be a function which belongs to $F(L, M)^{\mathcal{H}_k}$. From the definition of the propagation operator, we see that $(Gf)(x) = T_{w_x}^{-1} \cdot f(w_x x)$.

Suppose that $x \in L$ and $1 \leq i < k$. If $a_i(x) > 0$, we have $T_{w_{s_i x}} = T_{w_x} T_i$ by Proposition 2.2. Hence $(Gf)(s_i x) = T_i^{-1} \cdot (Gf)(x)$ because $w_{s_i x} s_i x = w_x x$. This also implies that $(Gf)(s_i x) = T_i \cdot (Gf)(x)$ if $a_i(x) < 0$.

Let us consider the case where $a_i(x) = 0$. Then we have $\sigma_x(i + 1) = \sigma_x(i) + 1$ and $T_i^{-1} T_{w_x}^{-1} = T_{w_x}^{-1} T_{\sigma_x(i)}^{-1}$ by Proposition 2.2. Now note that

$$a_{\sigma_x(i)}(w_x x) = \epsilon_{\sigma_x(i)}(w_x x) - \epsilon_{\sigma_x(i+1)}(w_x x) = \epsilon_i(x) - \epsilon_{i+1}(x) = a_i(x) = 0.$$

Hence we find that $T_{\sigma_x(i)}^{-1} \cdot f(w_x x) = (\rho(T_{\sigma_x(i)}^{-1})f)(w_x x) = f(w_x x)$ from Proposition 2.7. Therefore $T_i^{-1} \cdot (Gf)(x) = (Gf)(x)$. \square

For $1 \leq i < k$ and $\lambda \in V^*$, we define $Y_i(\lambda) \in \mathcal{H}_k$ by

$$Y_i(\lambda) := \frac{(e^{\lambda(v_i)} - e^{\lambda(v_{i+1})})T_i - e^{\lambda(v_i)}(\alpha e^{\lambda(v_{i+1})} + 1 - q)}{\alpha e^{\lambda(v_i + v_{i+1})} + e^{\lambda(v_i)} - q e^{\lambda(v_{i+1})}}.$$

Lemma 3.10. *The following equalities hold.*

- (i) $Y_i(s_i \lambda) Y_i(\lambda) = 1$ for $1 \leq i < k$ and $\lambda \in V^*$.
- (ii) $Y_{i+1}(s_i s_{i+1} \lambda) Y_i(s_{i+1} \lambda) Y_{i+1}(\lambda) = Y_i(s_{i+1} s_i \lambda) Y_{i+1}(s_i \lambda) Y_i(\lambda)$ for $1 \leq i \leq k - 2$ and $\lambda \in V^*$.

Proof. By a direct computation. \square

From Lemma 3.10, we obtain the following proposition.

Proposition 3.11. *Suppose that $\lambda \in V^*$. There exists a unique \mathbb{C} -algebra homomorphism $\phi_\lambda : \mathbb{C}[W] \rightarrow \mathcal{H}_k$ such that $\phi_\lambda(1) = 1$ and*

$$(3.7) \quad \phi_\lambda(s_i w) = Y_i(w\lambda) \phi_\lambda(w)$$

for $1 \leq i < k$ and $w \in W$.

Theorem 3.12. *Let M be a left \mathcal{H}_k -module. For $\lambda \in V^*$ and $m \in M$, we define a function $h_\lambda^m \in F(L, M)$ by*

$$(3.8) \quad h_\lambda^m(x) := \sum_{w \in W} e^{(w\lambda)(x)} (\phi_\lambda(w).m) \quad (x \in L),$$

where $.$ means the left action of \mathcal{H}_k on M . Then the function h_λ^m belongs to $F(L, M)^{\mathcal{H}_k}$ and is an eigenfunction of $\sum_{i=1}^k t_i$ with eigenvalue $\sum_{i=1}^k e^{-\lambda(v_i)}$. Therefore $G(h_\lambda^m)$ is an eigenfunction of $H|_{F_0(L, M)}$.

Proof. It is clear that h_λ^m is an eigenfunction of $\sum_{i=1}^k t_i$. From the relation (3.7), we have

$$\begin{aligned} T_j \phi_\lambda(w) &= \frac{e^{w\lambda(v_j)} (\alpha e^{w\lambda(v_{j+1})} + 1 - q)}{e^{w\lambda(v_j)} - e^{w\lambda(v_{j+1})}} \phi_\lambda(w) \\ &\quad + \frac{\alpha e^{w\lambda(v_j + v_{j+1})} + e^{w\lambda(v_j)} - q e^{w\lambda(v_{j+1})}}{e^{w\lambda(v_j)} - e^{w\lambda(v_{j+1})}} \phi_\lambda(s_j w) \end{aligned}$$

for $1 \leq j < k$ and $w \in W$. Moreover, from the definition of \widehat{I}_j , we see that

$$\widehat{I}_j(e^\mu) = \frac{\alpha e^{\mu(v_j + v_{j+1})} + (1 - q)e^{\mu(v_j)}}{e^{\mu(v_j)} - e^{\mu(v_{j+1})}} (e^\mu - e^{s_j \mu}) \quad (\mu \in V^*).$$

Combining the above equalities, we find that $\rho(T_i)h_\lambda^m = h_\lambda^m$ for $1 \leq i < k$. \square

Let M be a left \mathcal{H}_k -module. Set

$$\mathcal{F}(L_+, M) := \{f : L_+ \rightarrow M \mid T_i.f(x) = f(x) \text{ if } a_i(x) = 0\}.$$

We identify $F_0(L, M)$ with $\mathcal{F}(L_+, M)$ by the map $f \mapsto f|_{L_+}$ ($f \in F_0(L, M)$). Denote by H^+ the restriction of the discrete Hamiltonian H to $\mathcal{F}(L_+, M)$. Proposition 3.6 implies that the operator H^+ is given by

$$\begin{aligned} (3.9) \quad (H^+ f)(x) &= \sum_{a=1}^r \sum_{l=1}^{c_a} q^{c_a - l} T_{c_1 + \dots + c_{a-1} + l}^{-1} T_{c_1 + \dots + c_{a-1} + l + 1}^{-1} \cdots T_{c_1 + \dots + c_{a-1}}^{-1} f(x - v_{c_1 + \dots + c_a}) \\ &\quad - \frac{\alpha}{1 - q} \sum_{a=1}^r (c_a - [c_a]_q) f(x) \quad (f \in \mathcal{F}(L_+, M), x \in L_+), \end{aligned}$$

where (c_1, c_2, \dots, c_r) is the cluster coordinate of x .

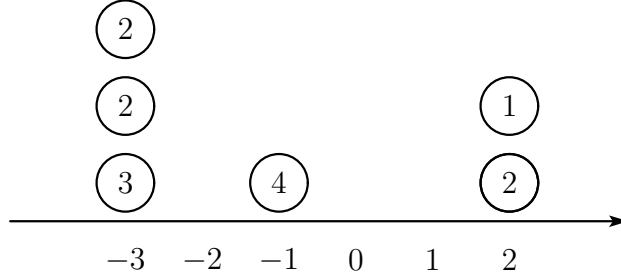


FIGURE 1.

Corollary 3.13. *Let M be a left \mathcal{H}_k -module. Suppose that $\lambda \in V^*$ and $m \in M$, and consider the function h_λ^m defined by (3.8). Then $h_\lambda^m|_{L_+}$ belongs to $\mathcal{F}(L_+, M)$ and is an eigenfunction of H^+ with eigenvalue $\sum_{i=1}^k e^{-\lambda(v_i)}$.*

Proof. It follows from Theorem 3.12 and the equality $G(h_\lambda^m)|_{L_+} = h_\lambda^m|_{L_+}$. \square

4. ALGEBRAIC CONSTRUCTION OF MULTI-SPECIES q -BOSON SYSTEM

4.1. Setting. In the rest of this article we fix a positive integer N . For a positive integer c , set

$$I_{N,c} := \{1, 2, \dots, N\}^c,$$

$$I_{N,c}^+ := \{(\mu_1, \dots, \mu_c) \in I_{N,c} \mid \mu_1 \leq \dots \leq \mu_c\}.$$

Let $x \in L_+$ and (c_1, \dots, c_r) be the cluster coordinate of x . For $\boldsymbol{\mu} \in I_{N,k}$, we define $\boldsymbol{\mu}[x] \in I_{N,k}$ as follows. According to the decomposition $I_{N,k} = I_{N,c_1} \times \dots \times I_{N,c_r}$, we write $\boldsymbol{\mu} = (\boldsymbol{\mu}_1, \dots, \boldsymbol{\mu}_r)$, where $\boldsymbol{\mu}_a \in I_{N,c_a}$ ($1 \leq a \leq r$). Then let $\boldsymbol{\mu}_a^+$ be the unique element of I_{N,c_a}^+ which is a rearrangement of $\boldsymbol{\mu}_a$. Now set $\boldsymbol{\mu}[x] := (\boldsymbol{\mu}_1^+, \dots, \boldsymbol{\mu}_r^+)$. For example, if $k = 5$, $x = 2v_1 + 2v_2 - v_3 - v_4 - v_5$ and $\boldsymbol{\mu} = (3, 1, 4, 2, 5)$, then $(c_1, c_2) = (2, 3)$, $\boldsymbol{\mu}_1 = (3, 1)$, $\boldsymbol{\mu}_2 = (4, 2, 5)$, and $\boldsymbol{\mu}[x] = (1, 3, 2, 4, 5)$.

Set

$$\mathcal{S} := \{(x, \boldsymbol{\nu}) \in L_+ \times I_{N,k} \mid \boldsymbol{\nu} = \boldsymbol{\nu}[x]\}.$$

We identify \mathcal{S} with the set of configurations of k bosonic particles of N species on the one-dimensional lattice \mathbb{Z} as follows. For $x = \sum_{i=1}^k m_i v_i \in L_+$ and $\boldsymbol{\nu} = (\nu_1, \dots, \nu_k)$, we assign to $(x, \boldsymbol{\nu})$ the configuration such that the particles with the color ν_1, \dots, ν_k are on the sites m_1, \dots, m_k , respectively. For example, if $k = 6$, $N = 4$, $x = 2v_1 + 2v_2 - v_3 - 3v_4 - 3v_5 - 3v_6$ and $\boldsymbol{\nu} = (1, 2, 4, 2, 2, 3)$, then $(x, \boldsymbol{\nu})$ corresponds to the configuration in Figure 1.

Denote the set of \mathbb{R} -valued functions on \mathcal{S} by $F(\mathcal{S})$. In the rest of this paper we construct the transition rate matrix $Q : F(\mathcal{S}) \rightarrow F(\mathcal{S})$ of a continuous-time Markov process on \mathcal{S} .

4.2. Derivation of the transition rate matrix. Hereafter we assume that

$$0 < q < 1.$$

Let $U = \bigoplus_{\mu=1}^N \mathbb{C}u_\mu$ be the N -dimensional vector space with the basis $\{u_\mu\}_{\mu=1}^N$. Consider the \mathbb{C} -linear operator $R \in \text{End}(U^{\otimes 2})$ defined by

$$R(u_\mu \otimes u_{\mu'}) = \begin{cases} q^{1/2}u_{\mu'} \otimes u_\mu & (\mu > \mu'), \\ u_\mu \otimes u_{\mu'} & (\mu = \mu'), \\ (1-q)u_\mu \otimes u_{\mu'} + q^{1/2}u_{\mu'} \otimes u_\mu & (\mu < \mu'). \end{cases}$$

Let $R_i \in \text{End}(U^{\otimes k})$ ($1 \leq i < k$) be the linear operator acting as R on the tensor product of the i -th and $(i+1)$ -th component of $U^{\otimes k}$.

Theorem 4.1. [7] *The assignment $T_i \mapsto R_i$ ($1 \leq i < k$) uniquely extends to a \mathbb{C} -algebra homomorphism $\mathcal{H}_k \rightarrow \text{End}(U^{\otimes k})$.*

In the following we regard $U^{\otimes k}$ as a left \mathcal{H}_k -module with respect to the action defined above.

For $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k) \in I_{N,k}$, we set

$$\mathbf{u}_\mu := u_{\mu_1} \otimes \cdots \otimes u_{\mu_k} \in U^{\otimes k},$$

and

$$\ell(\boldsymbol{\mu}) := \#\{(i, j) \mid 1 \leq i < j \leq k \text{ and } \mu_i > \mu_j\}.$$

Then any function $f : L_+ \rightarrow U^{\otimes k}$ is uniquely written in the form

$$(4.1) \quad f(x) = \sum_{\boldsymbol{\mu} \in I_{N,k}} q^{\ell(\boldsymbol{\mu})/2} f_\mu(x) \mathbf{u}_\mu \quad (x \in L_+),$$

where f_μ is a \mathbb{C} -valued function on L_+ . Then the space $\mathcal{F}(L_+, U^{\otimes k})$ has the following description.

Proposition 4.2. *In the above notation the following statements are equivalent.*

- (i) $f \in \mathcal{F}(L_+, U^{\otimes k})$.
- (ii) Suppose that $x \in L_+$ and $1 \leq i < k$. If $a_i(x) = 0$ then $f_{\dots, \mu_i, \mu_{i+1}, \dots}(x) = f_{\dots, \mu_{i+1}, \mu_i, \dots}(x)$ for all $\boldsymbol{\mu} = (\mu_1, \dots, \mu_k) \in I_{N,k}$.

Proof. It follows from the definition of the operator R and (4.1) by a direct computation. \square

We define the \mathbb{C} -linear map $\varphi : F(\mathcal{S}) \rightarrow \mathcal{F}(L_+, U^{\otimes k})$ by

$$(\varphi h)(x) := \sum_{\boldsymbol{\mu} \in I_{N,k}} q^{\ell(\boldsymbol{\mu})/2} h(x, \boldsymbol{\mu}[x]) \mathbf{u}_\mu \quad (h \in F(\mathcal{S}), x \in L_+).$$

Proposition 4.2 implies that the function defined by the right hand side above belongs to $\mathcal{F}(L_+, U^{\otimes k})$. The map φ is an isomorphism with the inverse

$$(\varphi^{-1}f)(x, \boldsymbol{\nu}) = f_\nu(x) \quad (f \in \mathcal{F}(L_+, U^{\otimes k}), (x, \boldsymbol{\nu}) \in \mathcal{S}),$$

where $f_{\boldsymbol{\nu}}$ is defined by (4.1).

Now let us write down the operator $\varphi^{-1}H^+\varphi : F(\mathcal{S}) \rightarrow F(\mathcal{S})$. For $1 \leq c \leq k$, we set

$$A^{(c)} := \sum_{l=1}^c q^{c-l} T_l^{-1} T_{l+1}^{-1} \cdots T_{c-1}^{-1} \in \mathcal{H}_k.$$

It acts on $U^{\otimes c}$. We set the matrix element $A_{\boldsymbol{\mu}, \boldsymbol{\mu}'}^{(c)} \in \mathbb{R}$ by

$$A^{(c)} \mathbf{u}_{\boldsymbol{\mu}} = \sum_{\boldsymbol{\mu}' \in I_{N,c}} A_{\boldsymbol{\mu}, \boldsymbol{\mu}'}^{(c)} \mathbf{u}_{\boldsymbol{\mu}'}.$$

Suppose that $h \in F(\mathcal{S})$ and $(x, \boldsymbol{\nu}) \in \mathcal{S}$. Denote by (c_1, \dots, c_r) the cluster coordinate of x , and decompose $\boldsymbol{\nu} = (\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_r)$, where $\boldsymbol{\nu}_a \in I_{N, c_a}^+$ for $1 \leq a \leq r$. From (3.9) we see that

$$(4.2) \quad \begin{aligned} (\varphi^{-1}H^+\varphi h)(x, \boldsymbol{\nu}) &= \sum_{a=1}^r \sum_{\boldsymbol{\eta} \in I_{N, c_a}} q^{\ell(\boldsymbol{\eta})/2} A_{\boldsymbol{\eta}, \boldsymbol{\nu}_a}^{(c_a)} (\varphi h)_{(\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_{a-1}, \boldsymbol{\eta}, \boldsymbol{\nu}_{a+1}, \dots, \boldsymbol{\nu}_r)} (x - v_{c_1 + \dots + c_a}) \\ &\quad - \frac{\alpha}{1-q} \sum_{a=1}^{c_a} (c_a - [c_a]_q) h(x, \boldsymbol{\nu}). \end{aligned}$$

Proposition 4.3. *Suppose that $1 \leq c \leq k$ and*

$$(4.3) \quad \boldsymbol{\nu} = (\underbrace{1, \dots, 1}_{m_1}, \dots, \underbrace{N, \dots, N}_{m_N}) \in I_{N,c}^+,$$

where m_1, \dots, m_N are non-negative integers satisfying $\sum_{i=1}^N m_i = c$. Then the matrix element $A_{\boldsymbol{\eta}, \boldsymbol{\nu}}^{(c)}$ is zero unless $\boldsymbol{\eta}$ is of the form

$$(4.4) \quad (\underbrace{1, \dots, 1}_{m_1}, \dots, \underbrace{b, \dots, b}_{m_b-1}, \dots, \underbrace{N, \dots, N}_{m_N}, b)$$

for some $1 \leq b \leq N$. If $\boldsymbol{\eta}$ is equal to (4.4), then

$$A_{\boldsymbol{\eta}, \boldsymbol{\nu}}^{(c)} = \frac{1 - q^{m_b}}{1 - q} q^{\sum_{i=b+1}^N m_i/2} = \frac{1 - q^{m_b}}{1 - q} q^{\ell(\boldsymbol{\eta})/2}.$$

Proof. For $n \geq 1$, let $(,)$ be the non-degenerate bilinear form on $U^{\otimes n}$ defined by

$$(u_{\mu_1} \otimes \cdots \otimes u_{\mu_n}, u_{\nu_1} \otimes \cdots \otimes u_{\nu_n}) = \delta_{\mu_1 \nu_1} \cdots \delta_{\mu_n \nu_n}.$$

Consider the linear operator $S \in \text{End}(U^{\otimes 2})$ defined by

$$S(u_{\mu} \otimes u_{\mu'}) = \begin{cases} (1 - q^{-1})u_{\mu} \otimes u_{\mu'} + q^{-1/2}u_{\mu'} \otimes u_{\mu} & (\mu > \mu'), \\ u_{\mu} \otimes u_{\mu} & (\mu = \mu'), \\ q^{-1/2}u_{\mu'} \otimes u_{\mu} & (\mu < \mu'). \end{cases}$$

Then it holds that $(R^{-1}\mathbf{u}, \mathbf{u}') = (\mathbf{u}, S\mathbf{u}')$ for $\mathbf{u}, \mathbf{u}' \in U^{\otimes 2}$. Let $S_i \in \text{End}(U^{\otimes k})$ ($1 \leq i < k$) be the linear operator acting as S on the tensor product of the i -th and $(i+1)$ -th component of $U^{\otimes k}$. Then we have

$$A_{\boldsymbol{\eta}, \boldsymbol{\nu}}^{(c)} = (A^{(c)}\mathbf{u}_{\boldsymbol{\eta}}, \mathbf{u}_{\boldsymbol{\nu}}) = \sum_{l=1}^c q^{c-l} (\mathbf{u}_{\boldsymbol{\eta}}, S_{c-1} \cdots S_l \mathbf{u}_{\boldsymbol{\nu}}).$$

From the definition of S , we see that

$$S_{c-1} \cdots S_l \mathbf{u}_{\boldsymbol{\nu}} = q^{-\sum_{i=b+1}^N m_i/2} \mathbf{u}_{\boldsymbol{\nu}^{(b)}},$$

where b is determined by the condition $m_1 + \cdots + m_{b-1} < l \leq m_1 + \cdots + m_b$ and $\boldsymbol{\nu}^{(b)}$ is the tuple (4.4). Therefore $A_{\boldsymbol{\eta}, \boldsymbol{\nu}}^{(c)} = 0$ unless $\boldsymbol{\eta}$ is of the form (4.4). If $\boldsymbol{\eta}$ is equal to (4.4), it holds that

$$\begin{aligned} A_{\boldsymbol{\eta}, \boldsymbol{\nu}}^{(c)} &= \sum_{l=m_1+\cdots+m_{b-1}+1}^{m_1+\cdots+m_b} q^{c-l-\sum_{i=b+1}^N m_i/2} \\ &= q^{\sum_{i=b+1}^N m_i/2} \sum_{l=m_1+\cdots+m_{b-1}+1}^{m_1+\cdots+m_b} q^{\sum_{i=1}^b m_i-l} = q^{\sum_{i=b+1}^N m_i/2} \frac{1-q^{m_b}}{1-q}. \end{aligned}$$

Here we used $c = \sum_{i=1}^N m_i$. This completes the proof. \square

For $\boldsymbol{\nu} \in I_{N,c}^+$ of the form (4.3) and $1 \leq b \leq N$, we set

$$\boldsymbol{\nu}^{b,\pm} := (\underbrace{1, \dots, 1}_{m_1}, \dots, \underbrace{b, \dots, b}_{m_b \pm 1}, \dots, \underbrace{N, \dots, N}_{m_N}) \in I_{N,c \pm 1}^+.$$

Let $(x, \boldsymbol{\nu}) \in \mathcal{S}$. Denote by (c_1, \dots, c_r) the cluster coordinate of x and write $\boldsymbol{\nu} = (\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_r)$ where $\boldsymbol{\nu}_a \in I_{N,c_a}^+$ ($1 \leq a \leq r$). For $(y, \boldsymbol{\eta}) \in \mathcal{S}$, we write $(x, \boldsymbol{\nu}) \rightsquigarrow (y, \boldsymbol{\eta})$ if the following conditions hold:

- (i) $y = x - v_{c_1+\cdots+c_a}$ for some $1 \leq a \leq r$.
- (ii) If $\epsilon_{c_1+\cdots+c_a}(x) - 1 > \epsilon_{c_1+\cdots+c_a+1}(x)$, then

$$\boldsymbol{\eta} = (\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_{a-1}, \boldsymbol{\nu}_a^{b,-}, b, \boldsymbol{\nu}_{a+1}, \dots, \boldsymbol{\nu}_r)$$

for some $1 \leq b \leq N$. If $\epsilon_{c_1+\cdots+c_a}(x) - 1 = \epsilon_{c_1+\cdots+c_a+1}(x)$, then

$$\boldsymbol{\eta} = (\boldsymbol{\nu}_1, \dots, \boldsymbol{\nu}_{a-1}, \boldsymbol{\nu}_a^{b,-}, \boldsymbol{\nu}_{a+1}^{b,+}, \boldsymbol{\nu}_{a+2}, \dots, \boldsymbol{\nu}_r)$$

for some $1 \leq b \leq N$.

Moreover, when the above conditions are satisfied, we set

$$(4.5) \quad c(x, \boldsymbol{\nu} | y, \boldsymbol{\eta}) := \frac{1-q^{m_b}}{1-q} q^{\sum_{i=b+1}^N m_i},$$

where m_i ($1 \leq i \leq N$) is the number of i in $\boldsymbol{\nu}_a$. We set $c(x, \boldsymbol{\nu} | y, \boldsymbol{\eta}) = 0$ unless $(x, \boldsymbol{\nu}) \rightsquigarrow (y, \boldsymbol{\eta})$.

Under the identification of \mathcal{S} with the set of configurations of k bosonic particles of N species given in the previous subsection, the condition $(x, \boldsymbol{\nu}) \rightsquigarrow (y, \boldsymbol{\eta})$ means

that the configuration corresponding to $(y, \boldsymbol{\eta})$ is obtained from that corresponding to $(x, \boldsymbol{\nu})$ by moving one particle to the left.

From (4.2) and the equality

$$\sum_{b=1}^N \frac{1 - q^{m_b}}{1 - q} q^{\sum_{i=b+1}^N m_i} = \frac{1 - q^{\sum_{i=1}^N m_i}}{1 - q},$$

we see that

$$\begin{aligned} (\varphi^{-1} H^+ \varphi h)(x, \boldsymbol{\nu}) &= \sum_{(y, \boldsymbol{\eta}) \in \mathcal{S}} c(x, \boldsymbol{\nu} | y, \boldsymbol{\eta}) \{h(y, \boldsymbol{\eta}) - h(x, \boldsymbol{\nu})\} \\ &+ \left(\frac{1 - q + \alpha}{1 - q} \sum_{a=1}^r [c_a]_q - \frac{\alpha}{1 - q} k \right) h(x, \boldsymbol{\nu}) \quad (h \in F(\mathcal{S}), (x, \boldsymbol{\nu}) \in \mathcal{S}), \end{aligned}$$

where (c_1, \dots, c_r) is the cluster coordinate of x . From the above formula we obtain the following result.

Theorem 4.4. *Set $\alpha = -(1 - q)$ and suppose that $0 < q < 1$. Then the operator*

$$Q := \varphi^{-1} H^+ \varphi - k$$

gives the transition rate matrix of a continuous-time Markov process on \mathcal{S} .

The resulting process is described as follows. In continuous time one particle may move from site i to $i - 1$ independently for each $i \in \mathbb{Z}$. The transition rate at which a particle with color b moves is given by the right hand side of (4.5), where m_i ($1 \leq i \leq N$) is the number of particles with color i in the cluster from which the moving particle leaves.

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